The bootstrap

Packages for this section

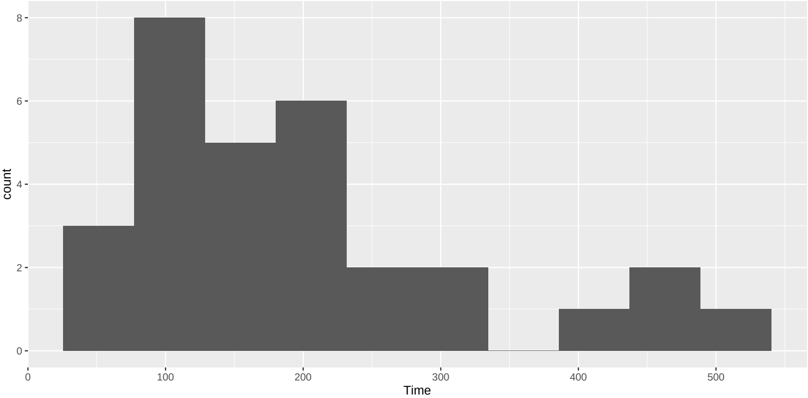
```
library(tidyverse)
library(bootstrap)
```

Source: Hesterberg et al (link)

Is my sampling distribution normal enough?

- Recall IRS data (used as a motivation for the sign test) :

```
ggplot(irs, aes(x=Time))+geom_histogram(bins=10)
```



- t procedure for the mean would not be a good idea because the distribution is skewed.

What *actually* matters

- It's not the distribution of the *data* that has to be approx normal (for a t procedure).
- What matters is the *sampling distribution of the sample mean*.
- If the sample size is large enough, the sampling distribution will be normal enough even if the data distribution is not.
    - This is why we had to consider the sample size as well as the shape.
- But how do we know whether this is the case or not? We only have *one* sample.

The (nonparametric) bootstrap

- Typically, our sample will be reasonably representative of the population.
- Idea: pretend the sample *is* the population, and sample from it *with replacement*.
- Calculate test statistic, and repeat many times.
- This gives an idea of how our statistic might vary in repeated samples: that is, its sampling distribution.
- Called the **bootstrap distribution** of the test statistic.
- If the bootstrap distribution is approx normal, infer that the true sampling distribution also approx normal, therefore inference about the mean such as t is good enough.
- If not, we should be more careful.

Why it works

- We typically estimate population parameters by using the corresponding sample thing: eg. estimate population mean using sample mean.
- This called **plug-in principle**.
- The fraction of sample values less than a value x called the **empirical distribution function** (as a function of x).
- By plug-in principle, the empirical distribution function is an estimate of the population CDF.
- In this sense, the sample *is* an estimate of the population, and so sampling from it is an estimate of sampling from the population.

Bootstrapping the IRS data

- Sampling with replacement is done like this (the default sample size is as long as the original data):

```
boot <- sample(irs$Time, replace=T)
mean(boot)
```

```
[1] 207.1667
```

- That's one bootstrapped mean. We need a whole bunch.

A whole bunch

- Use the same idea as for simulating power:

```
tibble(sim = 1:1000) %>%
  rowwise() %>%
  mutate(boot_sample = list(sample(irs$Time, replace = TRUE)))
```

```
# A tibble: 1,000 x 2
# Rowwise:
     sim boot_sample
   <int> <list>
 1     1 <dbl [30]>
 2     2 <dbl [30]>
 3     3 <dbl [30]>
 4     4 <dbl [30]>
 5     5 <dbl [30]>
 6     6 <dbl [30]>
 7     7 <dbl [30]>
 8     8 <dbl [30]>
 9     9 <dbl [30]>
10    10 <dbl [30]>
# i 990 more rows
```

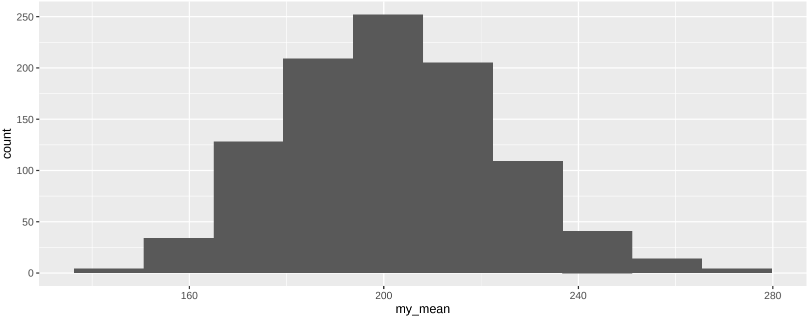
Get the mean of each of those

```
tibble(sim = 1:1000) %>%
  rowwise() %>%
  mutate(boot_sample = list(sample(irs$Time, replace = TRUE))) %>%
  mutate(my_mean = mean(boot_sample)) -> samples
samples
```

```
# A tibble: 1,000 x 3
# Rowwise:
     sim boot_sample my_mean
   <int> <list>        <dbl>
 1     1 <dbl [30]>     196 
 2     2 <dbl [30]>     202.
 3     3 <dbl [30]>     263.
 4     4 <dbl [30]>     173.
 5     5 <dbl [30]>     204.
 6     6 <dbl [30]>     197.
 7     7 <dbl [30]>     210.
 8     8 <dbl [30]>     160.
 9     9 <dbl [30]>     198.
10    10 <dbl [30]>     178.
# i 990 more rows
```

Sampling distribution of sample mean

```
ggplot(samples, aes(x=my_mean)) + geom_histogram(bins=10)
```

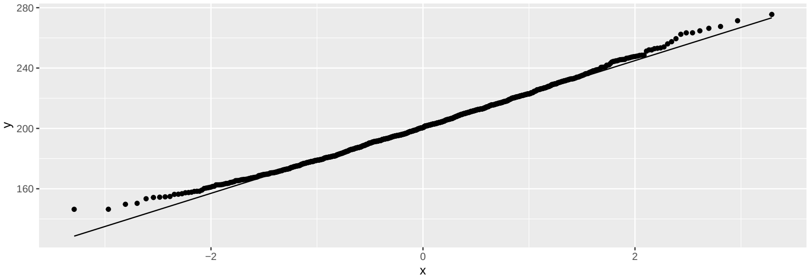


- Is that a slightly long right tail?

Normal quantile plot

might be better than a histogram:

```
ggplot(samples, aes(sample = my_mean)) +
  stat_qq()+stat_qq_line()
```



- a very very slight right-skewness, but very close to normal.

Confidence interval from the bootstrap distribution 1/2

There are two ways (at least). First way:

- percentile bootstrap interval: take the 2.5 and 97.5 percentiles (to get the middle 95%). This is easy, but not always the best:

```
samples %>%
  ungroup() %>% # undo the rowwise()
  reframe(ci = quantile(my_mean, c(0.025, 0.975))) %>%
  pull(ci) -> b_p
b_p
```

```
    2.5%    97.5%
162.5775 246.9092
```

Confidence interval from the bootstrap distribution 2/2

- Second way: bootstrap t: use the SD of the bootstrapped sampling distribution as the SE of the estimator of the mean and make a t interval:

```
n <- length(irs$Time)
t_star <- qt(0.975, n-1)
samples %>% ungroup() %>%
  summarize(boot_mean = mean(my_mean),
                      boot_sd = sd(my_mean)) %>%
  mutate(margin = t_star * boot_sd) %>%
  mutate(lo = boot_mean - margin,
         hi = boot_mean + margin) -> d
b_t <- c(d$lo, d$hi)
b_t
```

```
[1] 156.5070 246.4032
```

Comparing

- get ordinary t interval:

```
my_names=c("LCL", "UCL")
o_t <- t.test(irs$Time)$conf.int
```

- Compare the 2 bootstrap intervals with the ordinary t-interval:

```
tibble(limit=my_names, o_t, b_t, b_p)
```

```
# A tibble: 2 x 4
  limit   o_t   b_t   b_p
  <chr> <dbl> <dbl> <dbl>
1 LCL    155.  157.  163.
2 UCL    247.  246.  247.
```

- The bootstrap t and the ordinary t are very close
- The percentile bootstrap interval is noticeably shorter (common) and higher (skewness).

Which to prefer?

- If the intervals agree, then they are all good.
- If they disagree, they are all bad!
- In that case, use BCA interval (over).

Bias correction and acceleration

- this from “An introduction to the bootstrap”, by Brad Efron and Robert J. Tibshirani.
- there is way of correcting the CI for skewness in the bootstrap distribution, called the BCa method
- complicated (see the Efron and Tibshirani book), but implemented in `bootstrap` package.

Run this on the IRS data:

```r
bca=bcanon(irs$Time, 1000, mean)
bca$confpoints
```

```
     alpha bca point
[1,] 0.025  161.8333
[2,] 0.050  168.0667
[3,] 0.100  174.8333
[4,] 0.160  180.7667
[5,] 0.840  224.1333
[6,] 0.900  232.3000
[7,] 0.950  241.9333
[8,] 0.975  253.7333
```

use 2.5% and 97.5% points for CI

```r
bca$confpoints %>% as_tibble() %>%
  filter(alpha %in% c(0.025, 0.975)) %>%
  pull(`bca point`) -> b_bca
b_bca
```

```
[1] 161.8333 253.7333
```

Comparing

```
tibble(limit=my_names, o_t, b_t, b_p, b_bca)
```

```
# A tibble: 2 x 5
  limit   o_t   b_t   b_p b_bca
  <chr> <dbl> <dbl> <dbl> <dbl>
1 LCL    155.  157.  163.  162.
2 UCL    247.  246.  247.  254.
```

- The BCA interval says that the mean should be estimated even higher than the bootstrap percentile interval does.
- The BCA interval is the one to trust.

Bootstrapping the correlation

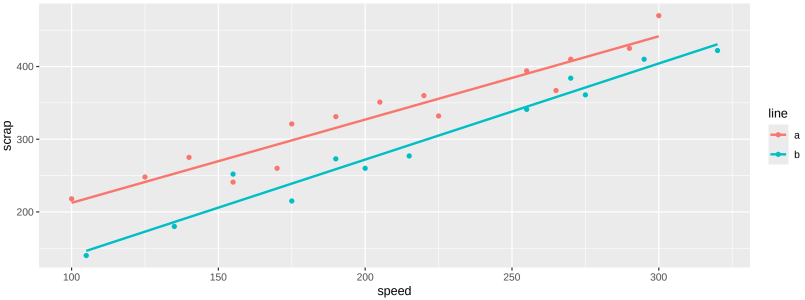
Recall the soap data:

```
url <- "http://ritsokiguess.site/datafiles/soap.txt"
soap <- read_delim(url," ")
soap
```

```
# A tibble: 27 x 4
    case scrap speed line
   <dbl> <dbl> <dbl> <chr>
 1     1   218   100 a
 2     2   248   125 a
 3     3   360   220 a
 4     4   351   205 a
 5     5   470   300 a
 6     6   394   255 a
 7     7   332   225 a
 8     8   321   175 a
 9     9   410   270 a
10    10   260   170 a
```

Scatterplot

```
ggplot(soap, aes(x=speed, y=scrap, colour=line))+
  geom_point()+geom_smooth(method="lm", se=F)
```



Comments

- Line B produces less scrap for any given speed.
- For line B, estimate the correlation between speed and scrap (with a confidence interval.)

Extract the line B data; standard correlation test

```
soap %>% filter(line=="b") -> line_b
with(line_b, cor.test(speed, scrap))
```

```

    Pearson's product-moment correlation

data:  speed and scrap
t = 15.829, df = 10, p-value = 2.083e-08
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
 0.9302445 0.9947166
sample estimates:
      cor
0.9806224
```

Bootstrapping a correlation 1/2

- This illustrates a different technique: we need to keep the x and y values *together*.
- Sample *rows* of the data frame rather than individual values of `speed` and `scrap`:

```
line_b %>% sample_frac(replace=T)
```

```
# A tibble: 12 x 4
    case scrap speed line
   <dbl> <dbl> <dbl> <chr>
 1    24   252   155 b
 2    22   260   200 b
 3    16   140   105 b
 4    25   422   320 b
 5    16   140   105 b
 6    19   341   255 b
 7    19   341   255 b
 8    19   341   255 b
 9    17   277   215 b
10    16   140   105 b
11    20   215   175 b
12    18   384   270 b
```

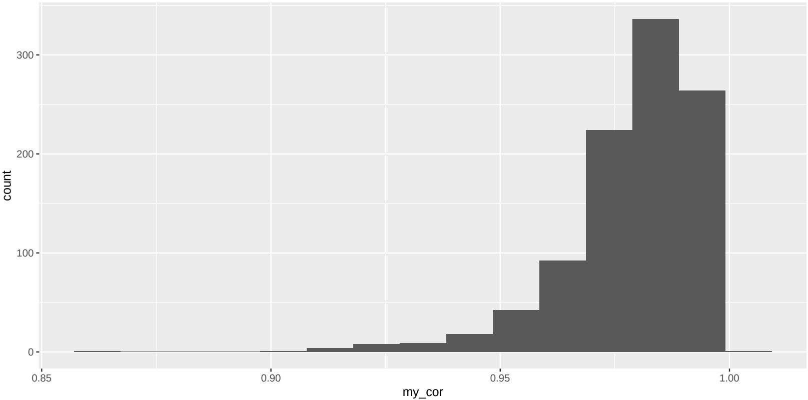
Bootstrapping a correlation 2/2

1000 times:

```
tibble(sim = 1:1000) %>%
  rowwise() %>%
  mutate(boot_df = list(sample_frac(line_b,
                                    replace = TRUE))) %>%
  mutate(my_cor =
           with(boot_df, cor(speed, scrap))) -> cors
```

A picture of this

```
ggplot(cors, aes(x=my_cor))+geom_histogram(bins=15)
```



Comments and next steps

- This is very left-skewed.
- Bootstrap percentile interval is:

```
cors %>% ungroup() %>%
  reframe(ci = quantile(my_cor, c(0.025, 0.975))) %>%
  pull(ci) -> b_p
b_p
```

```
     2.5%     97.5% 
0.9415748 0.9962462 
```

- We probably need the BCA interval instead.

Getting the BCA interval 1/2

- To use `bcanon`, write a function that takes a vector of row numbers and returns the correlation between `speed` and `scrap` for those rows:

```
theta <- function(rows, d) {
  d %>% slice(rows) %>% with(., cor(speed, scrap))
}
theta(1:3, line_b)
```

```
[1] 0.9928971
```

```
line_b %>% slice(1:3)
```

```
# A tibble: 3 x 4
   case scrap speed line
  <dbl> <dbl> <dbl> <chr>
1    16   140   105 b
2    17   277   215 b
3    18   384   270 b
```

- That looks about right.

Getting the BCA interval 2/2

- Inputs to `bcanon` are now:
  - row numbers (1 through 12 in our case: 12 rows in `line_b`)
  - number of bootstrap samples
  - the function we just wrote
  - the data frame:

```
points=bcanon(1:12, 1000, theta, line_b)$confpoints
points %>% as_tibble() %>%
  filter(alpha %in% c(0.025, 0.975)) %>%
  pull(`bca point`) -> b_bca
b_bca
```

```
[1] 0.9314334 0.9947799
```

Comparing the results

```
tibble(limit=my_names, o_c, b_p, b_bca)
```

```
# A tibble: 2 x 4
  limit   o_c   b_p b_bca
  <chr> <dbl> <dbl> <dbl>
1 LCL   0.930 0.942 0.931
2 UCL   0.995 0.996 0.995
```

- The bootstrap percentile interval doesn't go down far enough.
- The BCA interval seems to do a better job in capturing the skewness of the distribution.
- The ordinary confidence interval for the correlation is very similar to the BCA one, and thus seems to be trustworthy here even though the correlation has a very skewed distribution. (`cor.test` uses the Fisher z transformation which "spreads out" correlations close to 1).

The z-transformed bootstrapped correlations

```
cors %>%
  mutate(z = 0.5 * log((1+my_cor)/(1-my_cor))) %>%
  ggplot(aes(sample=z)) + stat_qq() + stat_qq_line()
```